Minimal metrics and the Abundance conjecture

Abundance conjecture

($\Delta = \mathbb{Q}$-divisor)

(X, Δ) klt projective pair,

assume $K_X + \Delta$ is nef

$$\Downarrow$$

$K_X + \Delta$ is semiample

(i.e. $m(K_X + \Delta)$ is basept free, for some $m \gg 0$)

Known:

- 3-folds : Miyaoka, Kawamata, Keel–Matsuki–McKernan (1980s–1990s)

- $K_X+\Delta\equiv 0$: Nakayama (2004)
- $K_X+\Delta$ is big : Shokurov, Kawamata (1985)
- if (X,Δ) satisfies Miyaoka's equality : Iwai, Matsumura, Müller (2024)

Higher dimensions (dim ≥ 4)

[L.-Peternell, Gongyo, Matsumura]

Assume MMP in Dimension $\leq \dim X - 1$.

(X,Δ) minimal klt pair.

Assume:

- $\chi(X,\mathcal{O}_X)\neq 0$
- $\exists\, \pi : Y\to X$ log resolution of (X,Δ) :

$\pi^*\mathcal{O}_X(K_X+\Delta)$ is hermitian semipositive

($\exists$ smooth metric h on $\pi^*\mathcal{O}_X(K_X+\Delta)$

s.t. $\Theta_h(K_X+\Delta) \geq 0$)

$\Rightarrow$ $K_X+\Delta$ is semiample.

Slogan: semipositive $\Rightarrow$ semiample

(if $\chi(X,\mathcal{O}_X) \neq 0$).

Q: where does the condition $\chi(X,\mathcal{O}_X)\neq 0$ come from?

A: maybe at the end of the talk.

More generally:

we don't need to assume semipositivity, we only need singular metrics with "generalised algebraic singularities"

we need to work with singular metrics

setup: $X =$ compact complex manifold

quasi-psh functions: these are locally sums of smooth and psh functions

Comparison of singularities

$\underline{\varphi_1} \preccurlyeq \underline{\varphi_2} \iff \exists$ constant C:

$\uparrow$ "less singular"

$$\underline{\varphi_2} \leq \underline{\varphi_1} + C$$

$\rightsquigarrow$ equivalence relation on quasi-psh functions:

$$\varphi_1 \simeq \varphi_2 \iff \varphi_1 \preccurlyeq \varphi_2 \preccurlyeq \varphi_1$$

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let α = smooth closed real $(1,1)$ form on X

$$PSH(X, \alpha) := \{ \varphi \text{ quasi-psh} \mid \underbrace{\alpha + dd^c\varphi}_{} \geq 0 \}$$

(in the sense of currents)

$[dd^c := \frac{i}{2\pi} \partial\bar\partial]$

$PSH(X,\alpha) \neq \emptyset \iff \{\alpha\} \in H^{1,1}_{BC}(X,\mathbb{R})$ is pseudoeffective

<u>THE POINT:</u> $\exists\ \varphi_{min} \in PSH(X,\alpha)$ which is minimal wrt singularities

<u>proof:</u>

$$V_\alpha := \{ \varphi \in PSH(X,\alpha) \mid \varphi \leq 0 \}$$

$$\varphi_{min} := \sup_{\varphi \in V_\alpha} \varphi$$

we can show: $\varphi_{min} \in V_\alpha$

<u>clearly:</u> φ_{min} is having minimal singularities. $\square$

this is an L^∞-condition

Remark: If $\varphi_1 \approx \varphi_2 \Rightarrow$

$$\mathcal{J}(\varphi_1) = \mathcal{J}(\varphi_2)$$

(φ quasi-psh function

$\Rightarrow \mathcal{J}(\varphi) =$ sheaf of germs of locally L^2-integrable functions on X wrt φ:

$\forall x \in X \; \exists U \ni x$:

$$\int_U \underbrace{|f|^2 e^{-2\varphi}}_{\text{in other words: } |\cdot|_h := |\cdot| e^{-\varphi}} dV_\omega < \infty \quad)$$

<u>Notation</u>: X compact complex manifold

T = closed positive $(1,1)$-current on X

$\Rightarrow \mathcal{J}(T)_{min}$ = multiplier ideal of any current with minimal singularities in $\{T\}$

$$\{T\} \in H^{1,1}_{BC}(X,\mathbb{R})$$

explanation:

$\{T\} \ni \alpha$ smooth real $(1,1)$ form

$\varphi_{min} \in PSH(X,\alpha)$

$$\Rightarrow \underbrace{\alpha + dd^c\varphi_{min}}_{T_{min}} \in \{T\}$$

$$\mathcal{J}(T)_{min} := \mathcal{J}(\varphi_{min})$$

Theorem A: Assume MMP in dim $\leq n-1$.

Let (X,Δ) be a klt minimal projective pair of dimension n. Let $\pi: Y \longrightarrow X$ be a log resolution of (X,Δ).

Write:

$$\underbrace{K_Y+\Delta_Y}_{\text{klt}} \sim_{\mathbb{Q}} \pi^*(K_X+\Delta) + E$$

$(\Delta_Y, E \geq 0$, have no common components$)$

Let A be an ample divisor on Y.

Assume that $\exists D \geq 0$ on Y (fictional) and a sequence of positive integers $\{m_\ell\}_{\ell\in\mathbb{N}}$ s.t. $m_\ell \xrightarrow[\ell\to\infty]{} \infty$ and

(*) $$\left\{\begin{array}{l} \mathcal{J}\left(\ell\left(K_Y+\Delta_Y+\frac{1}{m\ell}A\right)\right)_{\min} \\ \qquad\qquad \rotatebox[origin=c]{-90}{$\subseteq$} \\ \mathcal{J}\left(\ell\left(K_Y+\Delta_Y\right)\right)_{\min}\otimes\mathcal{O}_X(D) \end{array}\right.$$

Then: if $\chi(X,\mathcal{O}_X)\neq 0$ or if $\kappa(X, K_X+\Delta)\geq 0$, then $K_X+\Delta$ is semiample.

Remark 1: (*) follows from the Abundance conjecture

The point: the Abundance conjecture for $\chi(X,\mathcal{O}_X)\neq 0$ is equivalent to (*).

Remark 2: we always have

$$\mathcal{J}(\ell(K_Y+\Delta_Y))_{\min} \subseteq \mathcal{J}(\ell(K_Y+\Delta_Y+\tfrac{1}{m}A))_{\min}$$

So the assumption in Theorem A means that these multiplier ideals are asymptotically the same.

Remark 3: (*) is an example of asymptotic equisingular approximation

So: to prove Abundance when $\chi(X,\mathcal{O}_X)\neq 0$, one needs to show (*).

Supercanonical currents

these are currents with minimal singular singularities but they are not defined by an L^2-condition but by an "exponential L^2 condition"

(Narasimhan-Simha '68, Tsuji '07, Berman-Demailly 2012: versions of supercanonical currents

Definition/Lemma:

X compact complex manifold,
α smooth closed real $(1,1)$-form
st. $\{\alpha\} \in H^{1,1}_{BC}(X, \mathbb{R})$ is
pseudoeffective.

Let $\mathcal{S}_\alpha := \{ \varphi \in PSH(X, \alpha) \mid$
$$\int_X e^{2\varphi} dV_\omega \leq 1 \}$$

Set $\varphi_{\alpha,can} := \sup_{\varphi \in \mathcal{S}_\alpha} \varphi$.

(this is well-defined because all $\varphi \in \mathcal{S}_\alpha$ are uniformly bounded from above)

Then $\varphi_{\alpha,can} \in PSH(X,\alpha)$ and it is minimal.

<u>Theorem B</u>: Let (X,Δ) be a projective klt pair, $K_X+\Delta$ is pseudoeffective. Let $\pi : Y \to X$ be a log resolution; write

$$K_Y + \Delta_Y =_{\mathbb{Q}} \pi^*(K_X+\Delta) + E$$

($\Delta_Y, E \geq 0$, have no common components).

Let A be an ample divisor on Y.

Let $\alpha \in \{K_Y+\Delta_Y\}$, $\omega \in \{A\}$ smooth $(1,1)$ forms.

Then:

(a) $\varphi_{\alpha,\mathrm{can}} = \lim_{\varepsilon \to 0} \varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$

(b) each $\varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$ $(\varepsilon > 0)$ depends only on the global sections of multiples of $K_Y + \Delta_Y + \varepsilon A$.

$(\alpha + \varepsilon\omega \in \{K_Y + \Delta_Y + \varepsilon A\})$

+ some other properties depending on the MMP.

The most important ~~statement~~:

all $\varphi_{\alpha+\varepsilon\omega,\mathrm{can}}$ $(0 < \varepsilon \ll 1)$ are continuous away from the non-nef locus of $K_Y + \Delta_Y$!